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# Radiative corrections to coincidence experiments in high energy electron–electron and positron–electron scattering†

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**Abstract.** The theory of coincidence experiments in electron–electron and positron–electron scattering is considered in which the energies of both the scattered and recoil particles are defined to lie within prescribed narrow limits of their values for elastic scattering. Approximate formulae valid at very high energies, where nonlogarithmic terms may be neglected, are worked out. Both the clashing beam and the stationary target type of experiment are discussed, and in the latter case the effect of finite angular resolution in the detector system of the scattered particle is investigated. Numerical results are given for beam energies of 1000 $mc^2$. The resulting radiative corrections are much larger than in those types of experiment previously considered by Tsai and this provides the possibility of a more significant experimental test of the theory.

## 1. Introduction

The theory of radiative corrections to the Møller formula for electron–electron scattering and the corresponding Bhabha formula for positron–electron scattering was first worked out by the present author (Redhead 1953). The results were independently confirmed by Polovin (1956). In this early work of Redhead and Polovin it was assumed that the energy of photons emitted during the collisions was subject to isotropic limitation as viewed from the laboratory or centre of mass (CM) frame. It was recognized that for comparison with results obtained from any particular experimental arrangement, an additional contribution to the cross section would have to be worked out arising from real photons with energies lying between some suitably chosen lower spherical limit $k_0$ and an upper limit $k_{\max}$ which would have an angular dependence depending on the particular experiment.

Thus we write the differential cross section for scattering as

$$\mathrm{d}\sigma = \mathrm{d}\sigma_1 + \mathrm{d}\sigma_2$$

where $\mathrm{d}\sigma_1$ is the result obtained by Redhead and Polovin involving the radiation of photons with energies less than $k_0$, and $\mathrm{d}\sigma_2$ is the contribution from photons with energies between $k_0$ and $k_{\max}$ ($k_0$, of course, cancels out from the final result).

The calculation of $\mathrm{d}\sigma_1$ involves the removal of the ultraviolet and infrared divergences as described in detail in the papers referred to above.

The first attempt to calculate $\mathrm{d}\sigma_2$ for a particular experimental set-up was made by Tsai (1960) who considered the case of electron–electron scattering where (i) the struck

† Based on part of a thesis submitted to the University of London for the degree of PhD.

particle was at rest and the scattered particle was alone observed with its energy defined to lie within suitable narrow limits of the value for elastic scattering, and (ii) two beams in which the particles moved with equal and opposite momenta were in collision, the emerging particles being observed in coincidence, with their directions defined to lie within appropriate narrow limits of those for elastic scattering, but with no observation of the energies of either of the emerging particles. In both these cases the experimental arrangement did not preclude the radiation of hard photons in certain directions, as Tsai was the first to stress. As a result the calculation of $d\sigma_2$ becomes very complicated, because we can no longer neglect the photon momentum in the numerators of the rationalized matrix elements nor can we assume the validity of elastic kinematics in the calculation. Indeed Tsai only obtained specific results in the very high energy limit, where nonlogarithmic terms could be neglected. Furthermore, the resulting radiative corrections were rather small even at 500 MeV beam energy, due to the fact that the contribution from the hard photons is of opposite sign to the rest of the radiative correction, thus producing partial cancellation in the final result.

The purpose of the present paper is to consider a type of coincidence experiment in which the theoretical difficulties encountered by Tsai would not occur. In order to ensure that no hard photons can be radiated in any direction it is necessary to suppose that both the emerging particles, detected in coincidence, have their energies restricted, by suitable energy analysis of the emerging beams, to lie within narrow prescribed limits of the values for elastic scattering. In this way appropriate soft photon approximations can be used in calculating $d\sigma_2$, and, in addition, the resulting corrections are much larger, due essentially to the greater limitation of the phase space available to the emitted photons.

We have worked out formulae valid at very high energies, directly comparable with those of Tsai, and have considered the cases of both the clashing beam type of experiment and the stationary target type of experiment. In the latter case we have considered separately the situations in which the variation in energy of the elastically scattered particle across the entrance slit of the detector of the scattered particle is either very large or very small compared with the energy resolution of the detector. This complication, first recognized by Tsai, does not occur in the clashing beam type of experiment, since in this case the energy of the elastically scattered particle does not vary with the angle of scattering.

In conclusion, we have illustrated the sort of results that can be obtained with beam energies of 1000 $mc^2$.

## 2. High energy coincidence cross sections

We consider first the case of electron–electron scattering in which one electron is scattered into a fixed element of solid angle $d\Omega$ defined by a rectangular entrance slit for the detector system of the scattered particle, while the recoil electron is detected through a second rectangular slit of suitably adjusted size, both the electrons having their energies analysed magnetically before final detection in coincidence.

At sufficiently high energies, where we can neglect unity compared with $\ln E$, $E$ being the CM energy in the collision in units of $mc^2$, we can use the Tsai approximation† (Tsai 1960) for the soft photon cross section, with anisotropic emission.

† Detailed investigation of the validity of this approximation has been given by Redhead (1970). Tsai actually retained some nonlogarithmic terms in his results, which is inconsistent, and have been omitted in our work.

We write, then, for the differential coincidence cross section

$$d\sigma_{\text{coinc}}^{\text{e–e}} \simeq d\sigma_0\left[1+\frac{8\alpha}{\pi}\left\{\frac{11}{12}\ln E-\left(\ln(E\sin\theta)-\frac{1}{2}\right)\ln\left(\frac{E}{K}\right)\right\}\right] \tag{1}$$

where $\theta$ is the angle of scattering in the CM frame, and

$$K = (k_{\max}(\boldsymbol{p}_1)k_{\max}(\boldsymbol{p}'_1)k_{\max}(\boldsymbol{p}_2)k_{\max}(\boldsymbol{p}'_2))^{1/4}. \tag{2}$$

$\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ denote the momenta of the incoming particles referred to the CM frame, and $\boldsymbol{p}'_1$ and $\boldsymbol{p}'_2$ the momenta of the outgoing particles referred to the same frame, in the elastic collision. $k_{\max}(\boldsymbol{p}_1)$ is the maximum photon energy that can be carried off by a photon emitted parallel to $\boldsymbol{p}_1$, as seen from the CM frame, similarly for the remaining factors under the root sign in (2). $d\sigma_0$ refers to the cross section for scattering given by second order perturbation theory, that is, the Møller cross section for scattering into the element of solid angle $d\Omega$, and $\alpha$ is the usual fine structure constant.

In what follows we shall always refer to the particle emerging at the angle $\theta$ as the scattered particle, the other emerging particle being referred to as the recoil particle.

The size of the entrance slit for observing the recoil particle must be chosen so as not to eliminate from observation any scattering events that would be otherwise admissible from the energy restriction point of view. Appropriate criteria for achieving this will be derived subsequently for the different types of experiment to be discussed.

Corresponding to equation (1), the result for positron–electron coincidence scattering is easily found to be

$$d\sigma_{\text{coinc}}^{\text{p–e}} \simeq d\sigma_0\left[1+\frac{8\alpha}{\pi}\left\{\frac{11}{12}\ln E-\left(\ln\left(2E\tan\frac{1}{2}\theta\right)-\frac{1}{2}\right)\ln\left(\frac{E}{K}\right)\right\}\right]. \tag{3}$$

$d\sigma_0$ now refers to the Bhabha cross section for positron–electron collisions in second order.

Our task is thus to evaluate $K$ for the various types of experiment in which we are interested.

## 3. Clashing beam coincidence experiments

We consider first the case of the clashing beam experiment in which the CM frame coincides with the laboratory frame.

We define $\Delta E'_1 = E'_1 - E$ where $E'_1$ is the actual energy of the particle scattered at the angle $\theta$, referred to the CM frame, and $E$ is, as usual, the energy of this particle for the elastic collision. Similarly $\Delta E'_2 = E'_2 - E$ for the recoil particle.

We denote by $\beta$ the angle between $\boldsymbol{k}$ and $\boldsymbol{p}'_1$, $\boldsymbol{k}$ being the momentum of the photon emitted, relative to the CM frame.

If we assume that $\Delta E'_1 \ll E$, then the relationship between $k$, the photon energy, and $\Delta E'_1$ is generally of the form

$$k = \frac{-2\Delta E'_1}{1+(p/E)\cos\beta} \tag{4}$$

where $p$ is the magnitude of the momentum of the particles in the elastic collision in units of $mc$.

The simple formula (4) breaks down under conditions for which $k$ becomes much larger than $\Delta E_1'$. Under extreme relativistic (ER) conditions this will be the case as $\beta$ approaches $\pi$, that is, when the photon is emitted nearly parallel to $\boldsymbol{p}_2'$.

Similarly, the relationship between $k$ and $\Delta E_2'$ is, for $\Delta E_2' \ll E$, and omitting the neighbourhood of $\beta = 0$ under ER conditions (which corresponds to photon emission nearly parallel to $\boldsymbol{p}_1'$)

$$k = \frac{-2\Delta E_2'}{1-(p/E)\cos\beta}.$$

We notice at once that, since $k$ is necessarily positive, then $\Delta E_1'$ and $\Delta E_2'$ are essentially negative quantities. If we suppose the magnitude of these two quantities to be less than some specified limit $\epsilon$ ($\ll E$), we can write for the maximum values of $k$ permitted by the two energy restrictions, considering the situation at high energies for which $p/E \simeq 1$

$$^1k = \frac{2\epsilon}{1+\cos\beta} \tag{5}$$

and

$$^2k = \frac{2\epsilon}{1-\cos\beta}. \tag{6}$$

In figure 1 we sketch $^1k$ and $^2k$ as functions of $\beta$. Clearly the curve ABC determines the maximum photon energy permitted by restricting the energies of both the emerging particles.

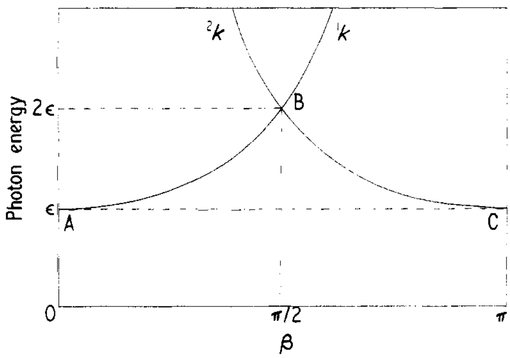


**Figure 1.** Restrictions on photon energy as a function of angle of emission for a coincidence experiment, in the CM frame.

Using (5) and (6) we can readily determine $k_{\max}(\boldsymbol{p}_1)$, etc. and hence find for $K$ in equation (2) the result

$$K = \begin{cases} \epsilon \sec \frac{1}{2}\theta & \text{for } 0 \leqslant \theta \leqslant \pi/2 \\ \epsilon \operatorname{cosec} \frac{1}{2}\theta & \text{for } \pi/2 \leqslant \theta \leqslant \pi. \end{cases} \tag{7}$$

Combining (7) with equations (1) or (3) gives the result for $d\sigma^{e-e}_{coinc}$ and $d\sigma^{p-e}_{coinc}$ respectively, for the clashing beam case.

The formulae are valid for

$$\epsilon \ll E \qquad \ln(\operatorname{cosec}\theta) \ll \ln E \qquad \ln E \gg 1.$$

It is easily shown that the maximum angle by which the direction of the recoil particle may differ from that obtaining in the elastic collision is $2\epsilon/E$. This sets a lower limit on the amount by which the angular aperture of the entrance slit for detecting the recoil particle must be increased, both in the plane of scattering and the transverse plane, over and above the size required to admit purely elastic collisions, in order that no permitted inelastic collisions may be excluded.

## 4. Stationary target coincidence experiments

### *4.1. Infinitesimal angular aperture in the plane of scattering for the detector system of the scattered particle*

We now define $\Delta\bar{E}'_1 = \bar{E}'_1 - \bar{E}'^{\,\mathrm{el}}_1$ where $\bar{E}'_1$ is the energy of the scattered particle referred to the laboratory frame, and $\bar{E}'^{\,\mathrm{el}}_1$ is the corresponding quantity in the elastic collision. Similarly $\Delta\bar{E}'_2 = \bar{E}'_2 - \bar{E}'^{\,\mathrm{el}}_2$ for the recoil particle.

In order to apply our formula for $K$, given in equation (2), we must first express $k$, the photon energy as seen from the CM frame, in terms of $\Delta\bar{E}'_1$ and $\Delta\bar{E}'_2$.

Denoting by $\gamma$ the angle between $\boldsymbol{k}$ and $\boldsymbol{p}_1$, a lengthy calculation yields the following rather simple results:

$$k = \frac{-2\Delta\bar{E}'_1/E}{(1+\cos\theta)\{1+(p/E)\cos\beta\}} \tag{8}$$

and

$$k = \frac{2\Delta\bar{E}'_2/E}{(1+\cos\theta)\{1+(p/E)\cos\beta\}-2\{1+(p/E)\cos\gamma\}}. \tag{9}$$

For a particular angle of emission these formulae apply only so long as $\Delta\bar{E}'_1$ and $\Delta\bar{E}'_2$ are sufficiently small to maintain $k \ll E$. As the angle of emission varies they will certainly break down when the denominators in (8) and (9) become sufficiently small. Under ER conditions this will occur near $\beta = \pi$ for (8), that is, when the photon is emitted nearly parallel to $\boldsymbol{p}'_2$, while for (9) the directions of photon emission near to which the formula fails, define a cone which passes through the direction of $\boldsymbol{p}'_1$ ($\beta = 0$, $\gamma = \theta$), but not through the direction of $\boldsymbol{p}'_2$, so that the two formulae never fail together.

We infer from (8) that $\Delta\bar{E}'_1$ is always negative, but, from (9), $\Delta\bar{E}'_2$ may have either sign, depending on the direction of emission of the photon.

If we denote the maximum permitted magnitude of $\Delta\bar{E}'_1$ and $\Delta\bar{E}'_2$ by $\bar{\epsilon}$, then the corresponding maximum values of $k$ permitted by the two restrictions are just

$${}^1k = \frac{2\bar{\epsilon}/E}{(1+\cos\theta)(1+\cos\beta)} \tag{10}$$

and

$${}^2k = \left|\frac{2\bar{\epsilon}/E}{(1+\cos\theta)(1+\cos\beta)-2(1+\cos\gamma)}\right| \tag{11}$$

where we have specialized to the case of ER conditions.

As we have just seen, these formulae do not apply near singularities in the indicated forms for ${}^1k$ and ${}^2k$, but where one formula breaks down the more stringent restriction is provided by the other formula, so that we can use (10) and (11) to determine correctly the maximum photon energy permitted by the double restriction in all cases.

We find then for $K$, in this case, the result

$$K = \begin{cases} \dfrac{\bar{\epsilon}}{E \sin\theta \cos\frac{1}{2}\theta} & \text{for } \theta \leqslant \psi \\ \dfrac{\bar{\epsilon}}{E \sin\theta}(1-\cos^4\frac{1}{2}\theta)^{-1/4} & \text{for } \theta \geqslant \psi \end{cases} \tag{12}$$

where $\psi = 2\cos^{-1}(\frac{1}{2})^{1/4}$.

Combining (12) with equations (1) or (3) gives the result for $d\sigma^{e-e}_{\text{coinc}}$ and $d\sigma^{p-e}_{\text{coinc}}$ respectively for the stationary target experiment with an infinitesimal entrance slit for observing the scattered particle.

The formulae are valid for

$$\bar{\epsilon} \ll E^2 \sin^2\theta \qquad \ln(\operatorname{cosec}\theta) \ll \ln E \qquad \ln E \gg 1.$$

In this case one can show that the angle by which the direction of the recoil particle deviates from the direction for elastic scattering is certainly less than $2(1+\sqrt{2})\bar{\epsilon}/E^3 \sin\theta(1-\cos\theta)$, and this sets a lower limit on the amount by which the angular aperture of the entrance slit for detecting the recoil particle must be increased above the size required for admitting purely elastic collisions, in this experiment.

The above theory has been worked out assuming for simplicity equal energy resolutions for the detectors of the scattered and recoil particles, but clearly the results could readily be adapted, if required, to the case where these energy resolutions were unequal.

### 4.2. *The effect of finite aperture in the detector system of the scattered particle*

Let us suppose, in a stationary target experiment, that the aperture of the detector of the scattered particle will admit particles scattered between angles $\bar{\theta}_0 - \Delta\bar{\theta}_0$ and $\bar{\theta}_0 + \Delta\bar{\theta}_0$, while the detector is adjusted to admit particles of energy within a range $\pm\bar{\epsilon}$ of the value for elastic scattering at the angle $\bar{\theta}_0$, that is, the energy range admitted by the detector of the scattered particle is

$$\bar{E}_1'^{\text{el}}(\bar{\theta}_0) - \bar{\epsilon} \leqslant \bar{E}_1' \leqslant \bar{E}_1'^{\text{el}}(\bar{\theta}_0) + \bar{\epsilon}. \tag{13}$$

Similarly for observation of the recoil particle we suppose that the energy range admitted by the detector is defined by

$$\bar{E}_2'^{\text{el}}(\bar{\theta}_0) - \bar{\epsilon} \leqslant \bar{E}_2' \leqslant \bar{E}_2'^{\text{el}}(\bar{\theta}_0) + \bar{\epsilon}. \tag{14}$$

The direction of the recoil particle is essentially unobserved.

We define $\Delta W = \bar{E}_1'^{\text{el}}(\bar{\theta}_0 - \Delta\bar{\theta}_0) - \bar{E}_1'^{\text{el}}(\bar{\theta}_0)$, so that the variation in energy of the elastically scattered particle across the aperture of the detector is sensibly $2\Delta W$.

For small $\Delta\bar{\theta}_0$, and assuming $(\pi - \bar{\theta}_0) \gg 1/E$, $E \gg 1$, it is easy to show that

$$\Delta W \simeq f(\bar{\theta}_0)\,\Delta\bar{\theta}_0$$

where (15)

$$f(\bar{\theta}_0) = 2E^3 \sin\theta_0 \cos^2\tfrac{1}{2}\theta_0.$$

$\theta_0$ is, of course, the scattering angle in the CM frame corresponding to the angle $\bar{\theta}_0$ in the laboratory frame.

Since $f(\bar{\theta}_0)$ is proportional to $E^3$, it is clear that at high energies the condition $\Delta W \ll \bar{\epsilon}$, which was tacitly assumed in the analysis of § 4.1, cannot in general be maintained.

We shall now proceed on the opposite assumption, namely, $\Delta W \gg \bar{\epsilon}$.

For any particular scattering angle $\bar{\theta}$ in the range $\bar{\theta}_0 - \Delta\bar{\theta}_0$ to $\bar{\theta}_0 + \Delta\bar{\theta}_0$ we can readily write down the photon energy required to change the energy of the scattered particle from the value $\bar{E}_1'^{\mathrm{el}}(\bar{\theta})$ to the value corresponding to the upper and lower limits of energy admitted by the detector, as defined in (13).

We write ${}^1k_{\mathrm{u}}$ and ${}^1k_{\mathrm{l}}$ for the photon energy referred to the CM frame, corresponding to these upper and lower limits respectively.

To compute ${}^1k_{\mathrm{u}}$, for example, we can use equation (8) and write

$$\Delta\bar{E}_1' = \bar{E}_1'^{\mathrm{el}}(\bar{\theta}_0) + \bar{\epsilon} - \bar{E}_1'^{\mathrm{el}}(\bar{\theta}) \simeq f(\bar{\theta}_0)[\bar{\theta} - \bar{\theta}_0 + \theta_1]$$

where the angle $\theta_1$ is defined by $\theta_1 = \bar{\epsilon}/f(\bar{\theta}_0)$. Then, at high energies

$${}^1k_{\mathrm{u}} = \frac{-2f(\bar{\theta}_0)[\bar{\theta} - \bar{\theta}_0 + \theta_1]}{E(1+\cos\theta)(1+\cos\beta)}.$$

Similarly

$${}^1k_{\mathrm{l}} = \frac{-2f(\bar{\theta}_0)[\bar{\theta} - \bar{\theta}_0 - \theta_1]}{E(1+\cos\theta)(1+\cos\beta)}.$$

In the same way we can use equation (9) to compute ${}^2k_{\mathrm{u}}$ and ${}^2k_{\mathrm{l}}$, the photon energies required to change the energy of the recoil particle from the value $\bar{E}_2'^{\mathrm{el}}(\bar{\theta})$ to the values corresponding to the upper and lower limits respectively, of the energy range admitted by the detector of the recoil particle, as specified in (14). Thus, at high energies, we find

$${}^2k_{\mathrm{u,l}} = \frac{2f(\bar{\theta}_0)[\bar{\theta}_0 - \bar{\theta} \pm \theta_1]}{E\{(1+\cos\theta)(1+\cos\beta) - 2(1+\cos\gamma)\}}.$$

The significance of the angle $\theta_1$ is that within the range of scattering angles $\bar{\theta}_0 - \theta_1 \leqslant \bar{\theta} \leqslant \bar{\theta}_0 + \theta_1$ the detector systems of the scattered and recoil particles will admit elastically scattered particles, while outside these limits only appropriate radiative collisions will be admitted.

As we have seen, in § 2, in the high energy approximation being used, the cross section depends on the maximum photon energy which can be radiated in four particular directions, namely, the directions of the vectors $\boldsymbol{p}_1$, $\boldsymbol{p}_1'$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_2'$. We can calculate values of ${}^1k_{\mathrm{u}}$, ${}^1k_{\mathrm{l}}$, ${}^2k_{\mathrm{u}}$ and ${}^2k_{\mathrm{l}}$ for these four directions of emission, which we distinguish by the subscript $s$, and write, in general, with $s = 1$, 2, 3 or 4

$$\begin{aligned}
{}^1k_{s\mathrm{u}} &= {}^1F_s f(\bar{\theta}_0)[\bar{\theta} - \bar{\theta}_0 + \theta_1] \\
{}^1k_{s\mathrm{l}} &= {}^1F_s f(\bar{\theta}_0)[\bar{\theta} - \bar{\theta}_0 - \theta_1] \\
{}^2k_{s\mathrm{u}} &= {}^2F_s f(\bar{\theta}_0)[\bar{\theta}_0 - \bar{\theta} + \theta_1] \\
{}^2k_{s\mathrm{l}} &= {}^2F_s f(\bar{\theta}_0)[\bar{\theta}_0 - \bar{\theta} - \theta_1]
\end{aligned} \tag{16}$$

where ${}^1F_s$ and ${}^2F_s$ are functions of $\theta$, which we approximate by their values at $\theta = \theta_0$, and list below:

$$
\begin{aligned}
{}^1F_1 &= \frac{-1}{2E\cos^4\frac{1}{2}\theta_0} & {}^2F_1 &= \frac{-1}{2E(1-\cos^4\frac{1}{2}\theta_0)} \\
{}^1F_2 &= \frac{-1}{2E\cos^2\frac{1}{2}\theta_0} & {}^2F_2 &= \infty \\
{}^1F_3 &= \frac{-2}{E\sin^2\theta_0} & {}^2F_3 &= \frac{2}{E\sin^2\theta_0} \\
{}^1F_4 &= \infty & {}^2F_4 &= \frac{-1}{2E\sin^2\frac{1}{2}\theta_0}.
\end{aligned}
$$

To determine the limits of photon energy permitted in any of the four cases distinguished by the subscript $s$, for any particular angle $\bar{\theta}$ in the range permitted by the aperture of the detector system of the scattered particle, we can conveniently plot, on a $(\bar{\theta}, k)$ diagram, the four quantities ${}^1k_{su}$, ${}^1k_{sl}$, ${}^2k_{su}$ and ${}^2k_{sl}$ as functions of $\bar{\theta}$, using (16). The results are four straight lines parallel in pairs with slopes determined by the factors ${}^1F_s$ and ${}^2F_s$. We must of course restrict ourselves to the physically significant upper half plane.

The results are sketched in figure 2. The photon energies permitted by the double energy restriction in the coincidence experiment are clearly restricted to lie within the triangle labelled ABC in figure 2($a$, $b$ and $d$), and within the trapezium ABCD in figure 2($c$). In particular, we notice that no scattering angles greater than $\bar{\theta}_0 + \theta_1$ are permitted at all, and only in figure 2($c$) is there a contribution from scattering angles less than $\bar{\theta}_0 - \theta_1$.

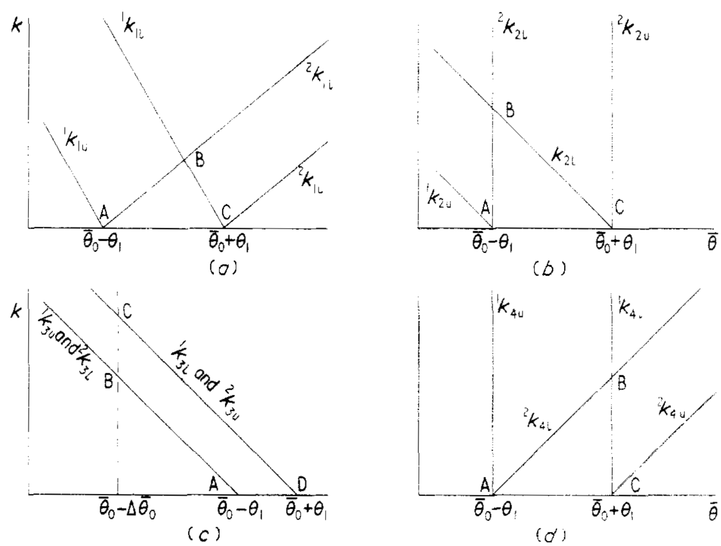


**Figure 2.** $(\bar{\theta}, k)$ diagrams. ($a$) Photon emitted parallel to $\boldsymbol{p}_1$; ($b$) photon emitted parallel to $\boldsymbol{p}'_1$; ($c$) photon emitted parallel to $\boldsymbol{p}_2$; ($d$) photon emitted parallel to $\boldsymbol{p}'_2$.

Furthermore, if the soft photon approximation is to apply, even in figure 2($c$), we must impose the restriction $\Delta W \ll E^2 \sin^2\theta_0$.

In order to compute the scattering cross section for the coincidence experiment, integrated over the scattering angle $\bar\theta$ from $\bar\theta = \bar\theta_0 - \Delta\bar\theta_0$ to $\bar\theta = \bar\theta_0 + \Delta\bar\theta_0$, a quantity which we denote by $Q_{\text{coinc}}$, we write

$$Q_{\text{coinc}} = \int_{\bar\theta_0 - \Delta\bar\theta_0}^{\bar\theta_0 + \Delta\bar\theta_0} \frac{d\sigma_{\text{coinc}}}{d\bar\theta}\, d\bar\theta. \tag{17}$$

For the differential cross section we can employ the formulae (1) or (3). These formulae assume a zero lower limit for permitted photon energy. For $\bar\theta$ in the range $\bar\theta_0 - \Delta\bar\theta_0$ to $\bar\theta_0 - \theta_1$ we can obtain the required differential cross section by taking the difference of the results appropriate to the upper and lower limits of permitted photon energy.

A straightforward but tedious calculation enables us to evaluate the integral in (17). Assuming $\theta_1 \ll \Delta\bar\theta_0$ (which is equivalent to $\Delta W \gg \bar\epsilon$), we obtain the results

$$Q^{\text{e–e}}_{\text{coinc}} \simeq 2\theta_1 \frac{d\sigma_0}{d\bar\theta}\bigg|_{\bar\theta=\bar\theta_0} \left[1 + \frac{8\alpha}{\pi}\left\langle \frac{11}{12}\ln E + \left\{\ln(E\sin\theta_0) - \frac{1}{2}\right\} \times \left\{\ln\left(\frac{(2\bar\epsilon/E^2)^{3/4}(\Delta W/E^2)^{1/4}}{\sin\theta_0}\right) - \frac{3}{4}\right\}\right\rangle\right] \tag{18}$$

$$Q^{\text{p–e}}_{\text{coinc}} \simeq 2\theta_1 \frac{d\sigma_0}{d\bar\theta}\bigg|_{\bar\theta=\bar\theta_0} \left[1 + \frac{8\alpha}{\pi}\left\langle \frac{11}{12}\ln E + \left\{\ln\left(2E\tan\frac{1}{2}\theta_0\right) - \frac{1}{2}\right\} \times \left\{\ln\left(\frac{(2\bar\epsilon/E^2)^{3/4}(\Delta W/E^2)^{1/4}}{\sin\theta_0}\right) - \frac{3}{4}\right\}\right\rangle\right]. \tag{19}$$

$\Delta W$ is itself a function of $\theta_0$ as shown in (15).

The validity of formulae (18) and (19) involves also a restriction on $\bar\epsilon$, namely, $\bar\epsilon \gg \Delta W/E^2 \sin^2\theta_0$, in order that the ER approximation $p/E \simeq 1$ may be applied in the analysis of figure 2(*c*).

The second order result expressed in the factor $2\theta_1\, d\sigma_0/d\bar\theta|_{\bar\theta=\bar\theta_0}$ is just what we would expect, since $2\theta_1$ is the range of angles over which purely elastic collisions are admitted in the coincidence experiment.

Denoting by $\Delta\bar\theta_0'$ the required semiangular aperture, in the plane of scattering, of the entrance slit for observing the recoil particle, so as to admit purely elastic scattering events, then it can be shown in the present case that the maximum angle by which the direction of the recoil particle differs from the direction for elastic scattering is just $2\Delta\bar\theta_0'$, and the size of the entrance slit must be adjusted accordingly.

## 5. Numerical results and discussion

To illustrate the sort of results that could be obtained in a particular experiment we consider the case of collisions with a beam energy of 1000 $mc^2$.

Considering first the stationary target type of experiment we have chosen a value for $\bar\epsilon$, the energy resolution of the detectors, as small as is reasonably possible, so as to produce the largest radiative correction. We have taken $\bar\epsilon = 2{\cdot}5\, mc^2$, which is 0·25 % of the incident beam energy. Taking the case of 90° scattering in the CM frame, the scattering angle in the laboratory frame is only 2° 33′. Clearly $\Delta\bar\theta_0$, the semiangular aperture of the detector of the scattered particle, must be kept as small as possible in relation to this sort of angle, if our assumption that the slowly-varying factors in the differential cross section can be approximated by their values at the midpoint of the scattering range, is to be valid. We have chosen $\Delta\bar{\theta}_0 = 7'\,40''$, which corresponds to $\Delta W = 25\,mc^2$ for 90° scattering. This value for $\Delta\bar{\theta}_0$ is approximately the same as was used in the experiment of Dally (1961) on 500 MeV electron–electron scattering.

In figure 3 we show the variation of $\Delta W$ with $\chi = \sin^2\frac{1}{2}\theta_0$, the fractional energy transfer in the collision, as determined from (15). (The simplified formula for the linear approximation to $\Delta W$ holds for $\chi$ as large as 0·99 at this energy, and indeed extrapolates correctly to the value zero at $\chi = 1$.)

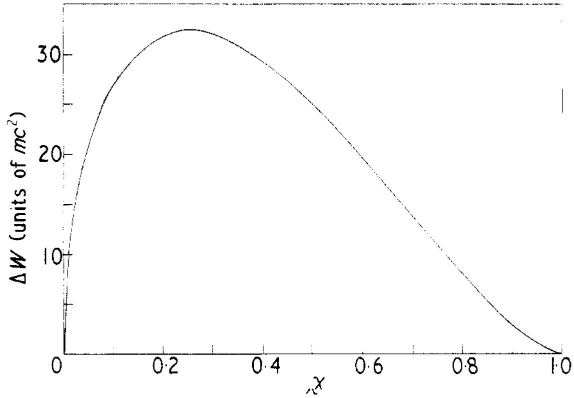


**Figure 3.** Variation of $\Delta W$ with fractional energy transfer $\chi$ in a coincidence experiment with stationary target. Incident laboratory energy $1000\,mc^2$. $\Delta\bar{\theta}_0 = 7'40''$.

Our approximations of assuming $\Delta W$ either much greater than or much less than $\bar{\epsilon}$ will certainly not be valid for values of $\chi$ between, say, 0·8 and 0·96, where $\Delta W$ becomes comparable with $\bar{\epsilon}$.

In figure 4 we show the radiative corrections as a function of $\chi$ for both electron–electron and positron–electron collisions, determined from equations (18) and (19) in the range $0{\cdot}01 < \chi < 0{\cdot}8$, and from equations (1) and (3), with $K$ determined from (12),

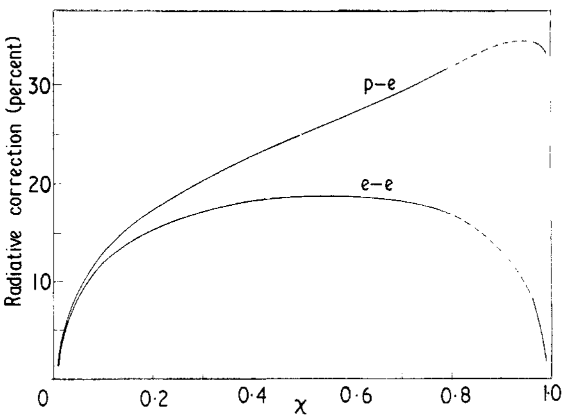


**Figure 4.** Radiative corrections for electron–electron and positron–electron scattering in a coincidence experiment with stationary target as a function of the fractional energy transfer $\chi$. Incident laboratory energy $1000\,mc^2$, energy resolution of detectors $2{\cdot}5\,mc^2$, $\Delta\bar{\theta}_0 = 7'40''$. Full curves, results predicted by formulae derived in text. Broken curves, speculative joining up of curves in two regions of approximation.

in the range $0{\cdot}96 < \chi < 0{\cdot}99$, the curves in the two ranges being joined by a speculative broken line. The radiative correction $\delta$ is defined simply by the equation $Q = Q_0(1-\delta)$, where $Q$ is the actual cross section and $Q_0$ is the cross section that would be predicted by second order perturbation theory.

These curves should give a reasonable idea of how the radiative corrections vary with $\chi$ for this experiment although as $\chi$ decreases below 0·2 or thereabouts the soft photon approximation will become increasingly inaccurate as the ratio $\Delta W/E^2 \sin^2\theta_0$ begins to rise significantly.

Referring to figure 4, we see that around 90° scattering in the CM frame ($\chi = 0{\cdot}5$) the correction for electron–electron scattering does not vary much with angle. For $\chi = 0{\cdot}5$, the correction for electron–electron scattering is 18·7%. This may be compared with the correction predicted at the same scattering angle by Tsai (1960), for the experiment of Dally (1961), of 5·5%. In Dally's experiment, of course, only the scattered electron had its energy analysed. Thus the effect of the coincidence in observation of scattered and recoil particle has been to increase the magnitude of the correction by a factor of more than three.

This clearly indicates that radiative corrections in a coincidence experiment would be much more easily susceptible of an experimental test than in the type of experiment conducted by Dally.

Turning to the case of a clashing beam type of experiment with beam energies of $1000\,mc^2$, still larger radiative corrections are to be expected, other factors being equal, due to the great increase in CM energy, as compared with the stationary target case. Indeed, if we take the energy resolution $\epsilon$ of the detectors in the clashing beam case as again equal to $2{\cdot}5\,mc^2$, the radiative corrections for 90° scattering, in electron–electron and positron–electron collisions respectively, would be 55·5% and 62·8%, as calculated using equations (1) and (3), with $K$ given by (7). With such large corrections higher order effects would, of course, modify the results considerably. The effect of processes involving the multiple emission of soft photons can be estimated by exponentiating the one-photon corrections we have calculated. Thus as the one-photon correction increases from 20% to 60%, the contribution of higher order terms would increase from an estimated 1·9% to as high as 14·9%.

In order to conduct a reasonable test of the theory with a clashing beam experiment at $1000\,mc^2$ energy, we must increase $\epsilon$ so as to reduce the magnitude of the corrections to a value which is large enough to be easily observed, but not so large that the effect of higher order processes becomes too significant. As an example we show in figure 5 the radiative corrections for both electron–electron and positron–electron collisions, taking $\epsilon = 100\,mc^2$. The discontinuity in the slope of the curves at $\chi = 0{\cdot}5$ occurs as one detector replaces the other in providing the more stringent restriction on the energy of photons emitted parallel to either of the incoming particles.

It should be noticed that the fact that the radiative corrections are not too large means that the counting rate for coincidence detection would be quite comparable with that obtaining in other types of experiment on Møller and Bhabha scattering, so that the statistical errors associated with the coincidence experiments would not be appreciably greater than those experienced in previous work.

To elucidate the significance of the results we have obtained, we may review briefly the more recent experimental tests of the radiative correction theory which have hitherto been made.

In Dally's experiment, already referred to, the experimental error was of the same order of magnitude as the radiative corrections, so that no significant test was possible.

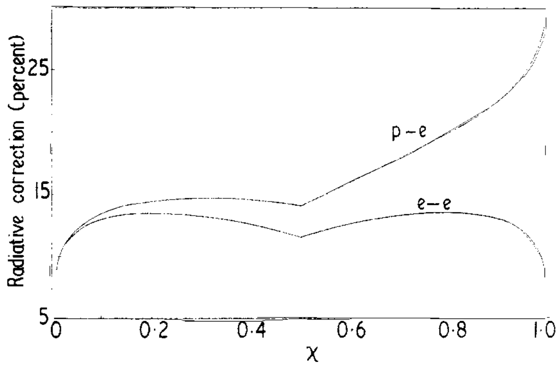


**Figure 5.** Radiative corrections for electron–electron and positron–electron scattering in a clashing beam coincidence experiment as a function of $\chi = \sin^2\frac{1}{2}\theta$. Incident beam energy $1000\,mc^2$, energy resolution of detectors $100\,mc^2$.

In order to increase the magnitude of the correction to be expected, Browman *et al* (1966) carried out an experiment on positron–electron scattering with an incident positron beam of energy 200 MeV and 500 MeV, and with scattering angles effectively in the neighbourhood of 180° in the CM frame. The theory for this experiment has been worked out in detail by Hearn *et al* (1969), and the results predicted are in good agreement with the experiment. Clearly the coincidence experiments we have been discussing provide the possibility of testing the theory over a much wider range of experimental conditions, than the very particular arrangement used by Browman and his co-workers.

In the case of the recent clashing beam experiments, such as those of Barber *et al* (1966) on electron–electron scattering and Augustin *et al* (1970) on positron–electron scattering, the results have generally been assessed by assuming the validity of the radiative correction theory and interpreting the residual discrepancy between experiment and theory in terms of possible form factors in the interaction between the particles.

It may be noticed that the effect of form factors decreases with energy in proportion to $E^2$, while the most rapidly varying term in the radiative correction behaves as $\ln^2 E$. In order to test the radiative corrections unambiguously in the clashing beam case, we should then consider experiments at lower beam energies, say 10 MeV, where the effect of form factors could be ignored, but in which, using the coincidence arrangement, reasonably large radiative corrections could be expected.

Finally we may remark that if an accurate low energy experiment were performed, it would be quite feasible to allow for the effect of nonlogarithmic terms, neglected in our approximate treatment. The exact theoretical treatment of soft photon problems, typified by the coincidence experiments discussed in the present work, is much simpler than in the case of hard photon problems, as, for example, in the work of Hearn *et al* referred to above.

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